On the existence of Killing vector fields*

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Abstract

In covariant metric theories of coupled gravity-matter systems the necessary and sufficient conditions ensuring the existence of a Killing vector field are investigated. It is shown that the symmetries of initial data sets are preserved by the evolution of hyperbolic systems.

1 Introduction

In any physical theory those configurations which possess symmetry are distinguished. They represent, for instance, the equilibrium states of the underlying systems. It might also happen that the relevant field equations are far too complicated not allowing us to have the general but only certain symmetric solutions of them. Einstein's theory of gravity provides immediate examples for these possibilities. In addition, in black hole physics the asymptotic final state of the gravitational collapse of an isolated object is expected to be represented by a stationary black hole solution (see e.g. [11, 19] for further details and references). It is also known that those solutions of Einstein's equations which possess symmetries do represent critical points of the phase space in the linear stability problem, i.e. in the investigation of the validity of the perturbation theory [13, 14, 1, 2, 3].

The above examples indicate that the identification of spacetimes possessing a symmetry is of obvious physical interest. In this paper the existence of symmetries will be studied in the case of coupled gravity-matter systems of the following types: The matter fields are assumed to satisfy suitable hyperbolic equations and to possess a minimal coupling to gravity. In addition, the Ricci tensor is supposed to be given as a function of the matter fields, their first covariant derivatives and the metric. Within this setting we address the following question: What are the necessary and sufficient conditions that can guarantee the existence of a Killing vector field so that the matter fields are also invariant?

The main purpose of the present paper is to answer this question. There is, however, a more pragmatic motivation beyond. We also would like to establish a detailed enough framework within which some of the claims of [17] concerning the existence of Killing vector fields within the characteristic initial value problem in the case of Einstein–Klein-Gordon, Einstein–[non-Abelian] Higgs or Einstein-[Maxwell]-Yang-Mills-dilaton systems (see [10] for analogous results concerning the Einstein-Maxwell case) can be justified.

*Dedicated to the memory of Professor Ágoston Bába

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Our main result is that hyperbolic evolutions preserve the symmetries of the initial data sets. In particular, initial symmetries are preserved in the case Einstein–Klein-Gordon, Einstein–[non-Abelian] Higgs and Einstein-[Maxwell]-Yang-Mills-dilaton systems.

The necessary and sufficient conditions ensuring the existence of a Killing vector field are given in terms of restrictions on certain initial data sets in an initial value problem associated with the considered system. There is only one significant requirement the applied initial value problem has to satisfy: The existence and uniqueness of solutions to quasilinear wave equations is expected to be guaranteed within its framework in the smooth setting. The initial value problems which satisfy this condition will be referred as 'appropriate initial value problems'. Immediate examples for appropriate initial value problems are the standard Cauchy problem (see e.g. Ref. [6]) and also in the characteristic initial value problem associated with an initial hypersurface represented by either the union of two smooth null hypersurfaces intersecting on a 2-dimensional spacelike surface [16, 18] or a characteristic cone [7, 4, 18]. Since no further requirement on the initial value problem is used anywhere in the derivation of our results we shall not make a definite choice among these appropriate initial value problems. However, as a direct consequence of this, our necessary and sufficient conditions cannot be as detailed as the corresponding restrictions applied e.g. in [5] where the case of Einstein-Maxwell systems in the framework of the standard Cauchy problem was considered. Although, the analogous investigations could be carried out case by case they are not attempted to be done here. Instead we introduce the following notation that will be applied on equal footing to any of the above mentioned appropriate initial value formulations: The initial data sets will be represented by the basic variables in square brackets while the initial hyper surface will always be denoted by Σ. Accordingly, whenever an 'initial data set' and an 'initial hypersurface' will be referred the relevant pair of these notions defined in either of the 'appropriate initial value problems' could be substituted.[1]

This paper is organized as follows: In section 2 we specify the gravity-matter systems to which our main results apply. Section 3 starts with the construction of a 'candidate' Killing vector field. Then a general procedure selecting the true Killing fields and also yielding the desired necessary and sufficient conditions is presented. Finally, in section 4 the particular case of Einstein-[Maxwell]-Yang-Mills-dilaton systems is considered.

Throughout this paper a spacetime (M, g_{ab}) is taken to be a smooth, paracompact, connected, orientable manifold M endowed with a smooth Lorentzian metric g_{ab}. Unless otherwise stated we shall use the notation and conventions of [19].

2 The gravity-matter systems

This section is to give a mathematically precise specification of the considered coupled gravity-matter systems.

The matter fields are considered to be represented by smooth tensor fields. Since the metric g_{ab} is assumed to be defined everywhere on M we may, without loss of generality, restrict considerations to tensor fields $T_{{}_{(i)}a...b}$ of type $(0, l_i)$ where $a...b$ denotes the $l_i(\in \mathbb{N} \cup \{0\})$ 'slots' of $T_{{}_{(i)}a...b}$. The matter fields might also have gauge dependence but, even if they have, the corresponding gauge or internal space indices will be suppressed. In most of the cases the spacetime indices of the matter field variables will also be suppressed and the relevant $(0, l_i)$ type tensor fields will simply be denoted

[1] All of the relevant field equations considered in this paper will (or in certain cases by making use of a 'hyperbolic reduction procedure' [8] will be shown to) possess the form of quasilinear wave equations. Thereby, in the particular case when our basic unknown is a function φ, one should think of the initial data set represented by $[\varphi]$ as a pair of the functions $\varphi|_\Sigma$ and $n^a\nabla_a\varphi|_\Sigma$, where n^a is normal to Σ, in the standard Cauchy problem while it consists of a single function $\varphi|_\Sigma$ in the characteristic initial value problem.

by $\mathcal{T}_{(i)}$. All the indices, however, will be spelled out explicitly in each of the not self explaining situations.

In our setting, just like in various other analogous investigations, the hyperbolic character of the matter field equations have more relevance than any further information related, for instance, to their particular form. Thereby, we shall assume that the matter fields $\mathcal{T}_{(i)}$ satisfy a quasi-linear, diagonal, second order hyperbolic system of the form [2]

$$\nabla^a\nabla_a\mathcal{T}_{(i)} = \mathcal{F}_{(i)}\left(\mathcal{T}_{(j)}, \nabla_c\mathcal{T}_{(j)}, g_{ef}\right), \tag{2.1}$$

where each of the $(0, l_i)$ type tensor fields $\mathcal{F}_{(i)}$ is assumed to be a smooth function of the indicated arguments (the obvious dependence on the points of M will be suppressed throughout this paper).

We assume, furthermore, that the matter fields are coupled to gravity so that the Ricci tensor R_{ab} can be given as a smooth function of the fields $\mathcal{T}_{(i)}$, their first covariant derivatives and the metric,

$$R_{ab} = R_{ab}\left(\mathcal{T}_{(i)}, \nabla_c\mathcal{T}_{(i)}, g_{ef}\right). \tag{2.2}$$

The last condition is satisfied, for instance, in Einstein's theory of gravity whenever the Lagrangian does contain at most first order derivatives of the matter field variables. Thereby, our results immediately apply to Einstein-matter systems of this kind. Note, however, that the above conditions are also satisfied by the 'conformally equivalent representation' of higher curvature theories possessing a gravitational Lagrangian that is a polynomial of the Ricci scalar (considered e.g. in Ref. [12]) and could also be satisfied by various other types of covariant metric theories of gravity.

The introduction of the above level of generality might seem to be pointless because even within Einstein's theory of gravity it has not been fixed in general whether such a coupled gravity-matter system possesses unique smooth solutions in suitable initial value formulations for smooth initial data sets. The remaining part of this section is to demonstrate – by making use of a straightforward adaptation of the 'hyperbolic reduction' procedure of Friedrich [8] to the present case – that (2.1) and (2.2) can be recast into the form of a system of coupled quasilinear wave equations and thereby they possess, up to diffeomorphisms, unique solutions in 'appropriate initial value problems' for the smooth setting.

Consider first the components of the metric g_{ab} and the matter fields $\mathcal{T}_{(i)}$ in some coordinates as the basic unknowns. Observe then that the Ricci tensor and $\nabla^a\nabla_a\mathcal{T}_{(i)}$ read in the associated local coordinates as

$$R_{\alpha\beta} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} + g_{\delta(\alpha}\partial_{\beta)}\Gamma^\delta + H'_{\alpha\beta}(g_{\varepsilon\rho}, \partial_\gamma g_{\varepsilon\rho}) \tag{2.3}$$

and

$$\nabla^\mu\nabla_\mu\mathcal{T}_{(i)} = g^{\mu\nu}\partial_\mu\partial_\nu\mathcal{T}_{(i)} - \sum_{k=1}^{l_i}\left(\mathcal{T}_{(i)}\right)^{[\alpha_k]}_{\ \delta}\left(R_{\alpha_k}{}^\delta + \partial_{\alpha_k}\Gamma^\delta\right) + \mathcal{H}'_{(i)}(g_{\varepsilon\rho}, \partial_\gamma g_{\varepsilon\rho}, \mathcal{T}_{(j)}, \partial_\gamma\mathcal{T}_{(j)}), \tag{2.4}$$

where ∂_α denotes the partial derivative operator with respect to the coordinate x^α, $\Gamma^\mu = g^{\alpha\beta}\Gamma^\mu{}_{\alpha\beta}$ with $\Gamma^\mu{}_{\varepsilon\rho}$ denoting the Christoffel symbol of $g_{\alpha\beta}$, moreover, $\left(\mathcal{T}_{(i)}\right)^{[\alpha_k]}_{\ \delta}$ stands for $T_{(i)\,\alpha_1\ldots\alpha_{k-1}\delta\alpha_{k+1}\ldots\alpha_{l_i}}$ and $H'_{\alpha\beta}$ and $\mathcal{H}'_{(i)}$ (the later with the suppressed indices $\alpha_1\ldots\alpha_{l_i}$) are appropriate smooth functions

[2] To be able to derive our later results, see equations (3.9) and (3.13), we need to keep our formalism in an explicitly covariant form. As opposed to the usual treatment of the matter fields in the initial value problems (see e.g. [19, 16, 18]) the matter field equations are assumed to be tensor equations. This, in particular, means that instead of requiring that the components of matter field variables, as a set of functions, satisfy hyperbolic wave equations involving only the 'flat wave operator' $g^{\mu\nu}\partial_\mu\partial_\nu$ exclusively (2.1) contains the complete action of the operator $g^{ab}\nabla_a\nabla_b$ on the matter field variables.

of the indicated variables. Thereby, (2.1) and (2.2) can be recast into the form

$$g^{\mu\nu}\partial_\mu\partial_\nu \mathcal{T}_{(i)} = \sum_{k=1}^{l_i} \left(\mathcal{T}_{(i)}\right)_{\delta}^{[\alpha_k]}\partial_{\alpha_k}\Gamma^\delta + \mathcal{H}_{(i)}(g_{\varepsilon\rho}, \partial_\gamma g_{\varepsilon\rho}, \mathcal{T}_{(j)}, \partial_\gamma \mathcal{T}_{(j)}) \tag{2.5}$$

$$g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} = 2g_{\delta(\alpha}\partial_{\beta)}\Gamma^\delta + H_{\alpha\beta}(g_{\varepsilon\rho}, \partial_\gamma g_{\varepsilon\rho}, \mathcal{T}_{(j)}, \partial_\gamma \mathcal{T}_{(j)}). \tag{2.6}$$

If the functions Γ^δ were known these equations would immediately give rise to a quasilinear, diagonal second order hyperbolic system for the unknowns and we would have done. Notice, however, that as in [8] Γ^δ can be replaced in the above equations with arbitrary 'gauge source functions' f^δ satisfying that $f^\delta = \Gamma^\delta$ on an arbitrarily chosen initial hypersurface Σ (where Γ^δ is always determined by the specified initial data $[g_{\alpha\beta}]$) and consider the relevant set of 'reduced equations' as propagation equations for the basic unknowns. From the corresponding unique solution the functions Γ^δ can be determined and, by refering to the twice contracted Bianchi identity, it can be demonstrated [8] (see also page 540 of [9]) that

$$\nabla^\mu\nabla_\mu(\Gamma^\delta - f^\delta) + R^\delta{}_\nu(\Gamma^\nu - f^\nu) = 0. \tag{2.7}$$

Recall then that $\Gamma^\delta = f^\delta$ on Σ and by (2.6) and its reduced form the entire initial data set for (2.7) has to vanish there. Thus we have that $\Gamma^\delta = f^\delta$ throughout the associated development, i.e. the solution of the reduced problem does satisfy the original set of coupled equations (2.5) and (2.6).

3 The construction of a Killing vector field

A spacetime admits a Killing vector field K^a whenever the Killing equation

$$\mathcal{L}_K g_{ab} = \nabla_a K_b + \nabla_b K_a = 0 \tag{3.1}$$

holds. Then we have that $X_{ab} = \nabla_a K_b$ is a 2-form field on M and its integrability condition $(dX)_{abc} = 0$ reads as

$$\nabla_a\nabla_b K_c + \nabla_c\nabla_a K_b + \nabla_b\nabla_c K_a = 0. \tag{3.2}$$

Replacing now $\nabla_a K_b$ with $-\nabla_b K_a$ (based on the antisymmetry of $\nabla_a K_b$) in the second term and using the definition of the Riemann tensor we get that

$$\nabla_a\nabla_b K_c + R_{bca}{}^d K_d = 0, \tag{3.3}$$

is equivalent to the integrability condition of X_{ab}. This equation holds for any Killing vector field on (M, g_{ab}). We will show below that its contraction

$$\nabla^a\nabla_a K_c + R_c{}^d K_d = 0, \tag{3.4}$$

which is a linear homogeneous wave equation for K^a, provides a mean to construct a 'candidate' Killing vector field. Clearly, any Killing vector field has to satisfy (3.4) but not all of its solutions will give rise to a Killing vector field. Our aim in this section to give the necessary and sufficient conditions on the initial data for (3.4), in terms of g_{ab} and $\mathcal{T}_{(i)}$, which guarantee that the corresponding solution of (3.4) will be a Killing vector field so that the matter fields will also be invariant, i.e. $\mathcal{L}_K \mathcal{T}_{(i)} = 0$.

The remaining part of this section is to show:

Theorem 3.1 *Let (M, g_{ab}) be a spacetime associated with a gravity-matter system specified in section 2. Denote by $D[\Sigma]$ the domain of dependence of an initial hypersurface Σ within an appropriate initial value problem. Then there exists a non-trivial Killing vector field K^a on $D[\Sigma]$, with*

$\mathcal{L}_K \mathcal{T}_{(i)} = 0$, *if and only if there exists a non-trivial initial data set* $[K^a]$ *for (3.4) so that* $[\mathcal{L}_K g_{ab}]$ *and* $[\mathcal{L}_K \mathcal{T}_{(i)}]$ *vanish*[3] *identically on* Σ.

Proof The necessity of the above conditions is trivial since the fields $\mathcal{L}_K g_{ab}$ and $\mathcal{L}_K \mathcal{T}_{(i)}$, along with their derivatives, vanish on Σ.

To see that the above conditions are also sufficient one can proceed as follows: Suppose that K^a satisfies (3.4) but otherwise it is an arbitrary vector field. It can be shown (by taking the covariant derivative of (3.4), commuting derivatives and applying the contracted Bianchi identity) that $\mathcal{L}_K g_{ab}$ satisfies the equation

$$\nabla^e \nabla_e \left(\mathcal{L}_K g_{ab}\right) = -2\mathcal{L}_K R_{ab} + 2R^e{}_{ab}{}^f \left(\mathcal{L}_K g_{ef}\right) + 2R^e{}_{(a}(\mathcal{L}_K g_{b)e}). \tag{3.5}$$

Moreover, by taking the Lie derivative of (2.2) we get[4]

$$\mathcal{L}_K R_{ab} = \sum_{(i)} \left(\frac{\partial R_{ab}}{\partial \mathcal{T}_{(i)}}\right) \mathcal{L}_K \mathcal{T}_{(i)} + \sum_{(i)} \left(\frac{\partial R_{ab}}{\partial \left(\nabla_e \mathcal{T}_{(i)}\right)}\right) \mathcal{L}_K \left(\nabla_e \mathcal{T}_{(i)}\right) + \left(\frac{\partial R_{ab}}{\partial g_{ef}}\right) \mathcal{L}_K g_{ef}. \tag{3.6}$$

Consider now the commutation relation of $\mathcal{L}_K$ and ∇_a

$$\mathcal{L}_K \left(\nabla_b \mathcal{T}_{(i)}\right) = \nabla_b \left(\mathcal{L}_K \mathcal{T}_{(i)}\right) - \sum_{k=1}^{l_i} \left(\mathcal{T}_{(i)}\right)^{[a_k]}_{e} \left[\nabla \mathcal{L}_K g\right]_{a_k}{}^{e}{}_{b}, \tag{3.7}$$

where $\left(\mathcal{T}_{(i)}\right)^{[a_k]}_{e}$ stands for $T_{(i)\, a_1 \dots a_{k-1} e a_{k+1} \dots a_{l_i}}$ and the notation

$$\left[\nabla \mathcal{L}_K g\right]_{a}{}^{c}{}_{b} = \frac{1}{2} g^{cf} \left\{\nabla_a \left(\mathcal{L}_K g_{fb}\right) + \nabla_b \left(\mathcal{L}_K g_{af}\right) - \nabla_f \left(\mathcal{L}_K g_{ab}\right)\right\} \tag{3.8}$$

has been used. Then, in virtue of (3.5), (3.6) and (3.7), $\mathcal{L}_K g_{ab}$ satisfies an equation of the form

$$\nabla^e \nabla_e \left(\mathcal{L}_K g_{ab}\right) = \mathrm{K}_{ab}(\mathcal{L}_K g_{cd}) + \mathrm{L}_{ab}\left(\nabla_c(\mathcal{L}_K g_{cd})\right) + \sum_{(i)} \mathrm{M}_{(i)\, ab}\left(\mathcal{L}_K \mathcal{T}_{(i)}\right) + \sum_{(i)} \mathrm{N}_{(i)\, ab}\left(\nabla_c(\mathcal{L}_K \mathcal{T}_{(i)})\right) \tag{3.9}$$

where K_{ab}, L_{ab}, $\mathrm{M}_{(i)\, ab}$ and $\mathrm{N}_{(i)\, ab}$ are linear and homogeneous functions in their indicated arguments.

Now we show that the same type of equation holds for $\mathcal{L}_K \mathcal{T}_{(i)}$. To see this consider first the Lie derivative of (2.1) with respect to the vector field K^a

$$\begin{aligned} \mathcal{L}_K \left(\nabla^e \nabla_e \mathcal{T}_{(i)}\right) &= \left(\mathcal{L}_K g^{ef}\right) \nabla_e \nabla_f \mathcal{T}_{(i)} + g^{ef} \mathcal{L}_K \left(\nabla_e \nabla_f \mathcal{T}_{(i)}\right) \qquad (3.10)\\ &= \sum_{(j)} \left(\frac{\partial \mathcal{F}_{(i)}}{\partial \mathcal{T}_{(j)}}\right) \mathcal{L}_K \mathcal{T}_{(j)} + \sum_{(j)} \left(\frac{\partial \mathcal{F}_{(i)}}{\partial \left(\nabla_c \mathcal{T}_{(j)}\right)}\right) \mathcal{L}_K \left(\nabla_c \mathcal{T}_{(j)}\right) + \left(\frac{\partial \mathcal{F}_{(i)}}{\partial g_{ab}}\right) \mathcal{L}_K g_{ab}. \end{aligned}$$

By applying the commutation relation (3.7) twice we get

$$\begin{aligned} \mathcal{L}_K \left(\nabla_e \nabla_f \mathcal{T}_{(i)}\right) &= \nabla_e \nabla_f \left(\mathcal{L}_K \mathcal{T}_{(i)}\right) - \left(\nabla_c \mathcal{T}_{(i)}\right) \left[\nabla \mathcal{L}_K g\right]_{f}{}^{c}{}_{e} \\ &- \sum_{k=1}^{l_i} \left[2 \left(\nabla_{(e|} \mathcal{T}_{(i)}\right)^{[a_k]}_{c} \left[\nabla \mathcal{L}_K g\right]_{a_k}{}^{c}{}_{|f)} + \left(\mathcal{T}_{(i)}\right)^{[a_k]}_{c} \nabla_e \left(\left[\nabla \mathcal{L}_K g\right]_{a_k}{}^{c}{}_{f}\right)\right]. \qquad (3.11) \end{aligned}$$

[3]As it will be shown below the 'evolution equations' for $\mathcal{L}_K g_{ab}$ and $\mathcal{L}_K \mathcal{T}_{(i)}$ are (3.9) and (3.13). By refering to these equations one can make immediate sense of the last part of our condition in either of the above mentioned appropriate initial value problems.

[4]Whenever $T_{a_1 \dots a_k}$ and $S_{b_1 \dots b_l}$ are tensor fields of type $(0,k)$ and $(0,l)$, respectively, $(\partial T_{a_1 \dots a_k}/\partial S_{b_1 \dots b_l})$ is considered to be a tensor field of type (l,k). Accordingly, the contraction $(\partial T_{a_1 \dots a_k}/\partial S_{b_1 \dots b_l})\mathcal{L}_K S_{b_1 \dots b_l}$ of $(\partial T_{a_1 \dots a_k}/\partial S_{b_1 \dots b_l})$ and $\mathcal{L}_K S_{b_1 \dots b_l}$ is a tensor field of type $(0,k)$.

The last term on the r.h.s. of (3.11) does not possess the needed form yet because it contains second order covariant derivatives of $\mathcal{L}_K g_{ab}$. Thereby it could 'get in the way' of the demonstration that $\mathcal{L}_K \mathcal{T}_{(i)}$ do satisfy quasilinear wave equations of the type (3.9). Note, however, that by the substitution $\mathcal{L}_K g_{ab} = \nabla_a K_b + \nabla_b K_a$, along with the commutation covariant derivatives and the application of the Bianchi identities, we get that whenever K^a satisfies (3.4)

$$g^{ef}\nabla_e\left([\nabla\mathcal{L}_K g]_{a_k}{}^{c}{}_{f}\right) = \frac{1}{2}g^{cd}\left\{\nabla^e\nabla_e\left(\mathcal{L}_K g_{a_k d}\right) - R_d{}^e\left(\mathcal{L}_K g_{a_k e}\right) + R_{a_k}{}^e\left(\mathcal{L}_K g_{ed}\right)\right\} \tag{3.12}$$

must also holds. Finally, by making use of all of the equations (3.6)–(3.12), it can be justified that $\mathcal{L}_K \mathcal{T}_{(i)}$ do really satisfy an equation of the form

$$\nabla^e\nabla_e\left(\mathcal{L}_K\mathcal{T}_{(i)}\right) = \mathcal{P}_{(i)}\left(\mathcal{L}_K g_{cd}\right) + \mathcal{Q}_{(i)}\left(\nabla_b(\mathcal{L}_K g_{cd})\right) + \sum_{(j)}\mathcal{R}_{(i)(j)}\left(\mathcal{L}_K\mathcal{T}_{(j)}\right) + \sum_{(j)}\mathcal{S}_{(i)(j)}\left(\nabla_c(\mathcal{L}_K\mathcal{T}_{(j)})\right) \tag{3.13}$$

where $\mathcal{P}_{(i)}$, $\mathcal{Q}_{(i)}$, $\mathcal{R}_{(i)(j)}$ and $\mathcal{S}_{(i)(j)}$ are linear and homogeneous functions in their indicated arguments.

To complete our proof consider a non-trivial initial data set $[K^a]$ on an initial hypersurface Σ associated with vanishing initial data, $[\mathcal{L}_K g_{ab}]$ and $[\mathcal{L}_K \mathcal{T}_{(i)}]$ for (3.9) and (3.13). Since (3.9) and (3.13) comprise a set of coupled linear and homogeneous wave equations for $\mathcal{L}_K g_{ab}$ and $\mathcal{L}_K \mathcal{T}_{(i)}$, which possesses the identically zero solution associated with a zero initial data set, we have that $\mathcal{L}_K g_{ab} \equiv 0$ and $\mathcal{L}_K \mathcal{T}_{(i)} \equiv 0$ throughout the domain where the associated unique solution of (3.4) does exist. To see, finally, that this domain has to coincide with $D[\Sigma]$ note that since (3.4) is also a linear homogeneous wave equation any solution of it can be shown – by making use of the 'patching together local solutions' techniques described e.g. on page 266 of [15] – to extend to the entire of the domain of dependence $D[\Sigma]$ of Σ. □

Note that the full set of equations (2.1), (2.2), (3.4), (3.9) and (3.13) could also be recast – by making use of the hyperbolic reduction techniques – into the form of a set of quasilinear wave equations. Moreover, the validity of the conditions of the above theorem can also be justified if the initial data $[g_{ab}]$ and $[\mathcal{T}_{(i)}]$, associated with (2.1) and (2.2), are given on Σ. Therefore in virtue of the above result we have

Corollary 3.1 *Denote by $D[\Sigma]$ the maximal development of an initial data set $[g_{ab}]$ and $[\mathcal{T}_{(i)}]$, associated with (2.1) and (2.2), specified on an initial hypersurface Σ within an appropriate initial value problem. Then there exists a non-trivial Killing vector field K^a on $D[\Sigma]$, with $\mathcal{L}_K \mathcal{T}_{(i)} = 0$, if and only if there exists a non-trivial initial data set $[K^a]$ for (3.4) so that the initial data, $[\mathcal{L}_K g_{ab}]$ and $[\mathcal{L}_K \mathcal{T}_{(i)}]$, for (3.9) and (3.13) vanish identically on Σ.*

Remark 3.1 *Remember that whenever only the existence of a Killing vector field ensured for the considered gravity-matter systems the matter fields need not to be invariant. Nevertheless, the non-invariance of them – represented by the fields $\mathcal{L}_K \mathcal{T}_{(i)}$ – has to 'develop' according to the linear homogeneous wave equations*

$$\nabla^e\nabla_e\left(\mathcal{L}_K\mathcal{T}_{(i)}\right) = \sum_{(j)}\left(\frac{\partial\mathcal{F}_{(i)}}{\partial\mathcal{T}_{(j)}}\right)\mathcal{L}_K\mathcal{T}_{(j)} + \sum_{(j)}\left(\frac{\partial\mathcal{F}_{(i)}}{\partial\left(\nabla_e\mathcal{T}_{(j)}\right)}\right)\nabla_e\left(\mathcal{L}_K\mathcal{T}_{(j)}\right) \tag{3.14}$$

and the constraints

$$0 = \sum_{(i)}\left(\frac{\partial R_{ab}}{\partial\mathcal{T}_{(i)}}\right)\mathcal{L}_K\mathcal{T}_{(i)} + \sum_{(i)}\left(\frac{\partial R_{ab}}{\partial\left(\nabla_e\mathcal{T}_{(i)}\right)}\right)\nabla_e\left(\mathcal{L}_K\mathcal{T}_{(i)}\right), \tag{3.15}$$

have also to be satisfied.

4 Einstein-[Maxwell]-Yang-Mills (E[M]YM) systems

In this section the case of matter fields for which the r.h.s. of (2.1) explicitly depends on second order derivatives of the matter field variables will be considered. However, instead of trying to have a result of 'maximal generality' we only illustrate what sort of modifications of the procedures of the previous section are needed to cover the particular case of E[M]YM systems which are described as follows.

A Yang-Mills field is represented by a vector potential A_a taking values in the Lie algebra $\mathfrak{g}$ of a Lie group G. For the sake of definiteness, throughout this paper, G will be assumed to be a matrix group and it will also be assumed that there exists a real inner product, denoted by (/), on $\mathfrak{g}$ which is invariant under the adjoint representation[5]. In terms of the gauge potential A_a the Lie-algebra-valued 2-form field F_{ab} is given as

$$F_{ab} = \nabla_a A_b - \nabla_b A_a + [A_a, A_b] \tag{4.1}$$

where [,] denotes the product in $\mathfrak{g}$. The Ricci tensor of an E[M]YM system is

$$R_{ab} = \chi \left\{ (F_{ae}/F_b{}^e) - \frac{1}{4} g_{ab} \left(F_{ef}/F^{ef} \right) \right\}, \tag{4.2}$$

with some constant χ, while the Yang-Mills field equations read as

$$\nabla^a F_{ab} + [A^a, F_{ba}] = 0. \tag{4.3}$$

By the substitution of the r.h.s. of (4.1) for F_{ab} into (4.3) and commuting derivatives we get

$$\nabla^a \nabla_a A_b = R_b{}^d A_d - [\nabla^a A_a, A_b] - [A_a, \nabla^a A_b] - [A^a, F_{ba}] + \nabla_b (\nabla^a A_a). \tag{4.4}$$

In virtue of (4.1), (4.3) and (4.4) we have that the considered E[M]YM system is almost of the type characterized in section 2. The only, however, significant difference is that on the r.h.s. of (4.4) we have the term $\nabla_b(\nabla^a A_a)$ which contains second order covariant derivatives of the gauge potential A_a. Fortunately, this term is of the same character as $\partial_\alpha \Gamma^\beta$ was in (2.6) and the same type of resolution does apply here. To see this substitute an arbitrary function $\mathcal{A}$, called 'gauge source function', for $\nabla^a A_a$ in the last term of (4.4) so that $\mathcal{A} = \nabla^a A_a$ holds on an initial hypersurface Σ. The yielded equation is referred to be the reduced form of (4.4). Then by the application of ∇^b to (4.4), and also to its reduced form, we get that

$$\nabla^b \nabla_b (\mathcal{A} - \nabla^a A_a) = 0. \tag{4.5}$$

In addition, we have that $\mathcal{A} = \nabla^a A_a$ on Σ, moreover, (4.4), along with its reduced form, implies that the initial data for (4.5) has to vanish identically on Σ. Thereby the unique solution of the reduced form of (4.4) is in fact a solution of (4.4) itself.

It follows from the above argument that an E[M]YM system differs from the systems specified in section 2 only in that in the present case we have to apply two types of gauge source functions f^δ and $\mathcal{A}$ and then use the subsidiary equations (2.7) and (4.5) to demonstrate that the unique smooth solutions of the 'reduced equations' – which equations are obviously the needed type of quasilinear wave equations – are, up to 'gauge transformations', unique solutions of the full E[M]YM equations, as well.

Turning back to the problem of the existence of a Killing vector field in the case of an E[M]YM system consider first a vector field K^a satisfying (3.4) and suppose that $\mathcal{L}_K(\nabla^a A_a)$ does vanish

[5] Note that the inner product (/) is not needed to be assumed to be positive definite as opposed to its use in [17].

throughout. Then, from (4.4) we get that an equation of the type (3.13) has to be satisfied by $\mathcal{L}_K A_a$. Thereby, a statement analogous to that of Theor.3.1 can immediately be recovered in the case of E[M]YM systems whenever a gauge potential A_a with $\mathcal{L}_K(\nabla^a A_a) = 0$ does exist.

Suppose now that a non-trivial initial data set $[K^a]$ for (3.4) is given so that $[\mathcal{L}_K g_{ab}]$ and $[\mathcal{L}_K A_a]$ vanish on an initial hypersurface Σ. We show that then there exists a gauge potential A^*_a (at least in a sufficiently small neighbourhood $\mathcal{O}$ of Σ) so that $[A^*_a]$ coincides with $[A_a]$ on Σ and $\mathcal{L}_K(\nabla^a A^*_a) = 0$. To see this replace A_a by a gauge equivalent potential

$$A^*_a = m^{-1}\left(\nabla_a m + A_a m\right), \tag{4.6}$$

where m is an arbitrary smooth G-valued function (defined in general locally on an open subset of M). Since $\mathcal{L}_K(\nabla^a A_a) = 0$ on Σ it is reasonable to require $[A^*_a]$ to coincide with $[A_a]$ on Σ. Accordingly, we set $m = \mathbb{E}$ (and if it is needed $\nabla_a m = 0$) on Σ, where $\mathbb{E}$ is the unit element of G. Since, so far only the initial data $[A^*_a]$ has been specified, we still have the freedom to choose the gauge source function $\mathcal{A}^* = \nabla^a A^*_a$ arbitrarily. Thereby we assume that $\nabla^a A^*_a$ is so that $\mathcal{L}_K(\nabla^a A^*_a)$ vanishes identically in $D[\Sigma]$. In addition, the generator of the gauge transformation (4.6) has to satisfy

$$\nabla^a\nabla_a m = m(\nabla^a A^*_a) - (\nabla^a A_a)m - A_a\nabla^a m + (\nabla^a m)m^{-1}(\nabla_a m + A_a m). \tag{4.7}$$

By making use of this equation m and, in turn, the desired gauge potential A^*_a can be constructed. Clearly, (4.7), with given 'source terms' $\nabla^a A^*_a$ and $\nabla^a A_a$, possesses (at least in a sufficiently small neighbourhood $\mathcal{O}$ of the initial hypersurface Σ) a unique solution so that $m = \mathbb{E}$ (and $\nabla_a m = 0$) on Σ. Moreover, for the gauge potential A^*_a, determined by this unique solution m via (4.6), $[\mathcal{L}_K A^*_a]$ vanishes on Σ and $\mathcal{L}_K(\nabla^a A^*_a) = 0$ throughout $\mathcal{O}$.

The following sums up what have been proven

Theorem 4.1 *Let (M, g_{ab}) be a spacetime associated with an E[M]YM system as it was specified above. Let, furthermore, Σ be an initial hypersurface within an appropriate initial value problem. Then there exist a non-trivial Killing vector field K^a and a gauge potential A^*_a so that $\mathcal{L}_K A^*_a = 0$ in a neighbourhood $\mathcal{O}$ of Σ, if and only if there exists a non-trivial initial data set $[K^a]$ for (3.4) so that $[\mathcal{L}_K g_{ab}]$ and $[\mathcal{L}_K A_a]$ vanish identically on Σ.*

Remark 4.1 *Combining now all the above results a statement analogous to that of Theor.4.1 can be proven for appropriate couplings (based on terms containing sufficiently low order derivatives of the matter fields variables) of Yang-Mills – matter systems in Einstein theory of gravity. In particular, it can be shown that Theor.4.1 generalizes to the case of Einstein-[Maxwell]-Yang-Mills-dilaton systems given by the Lagrangian*

$$\mathcal{L} = R + 2\nabla^e\psi\nabla_e\psi - e^{2\gamma_d\psi}(F_{ef}/F^{ef}), \tag{4.8}$$

where R is the Ricci scalar, ψ denotes the (real) dilaton field and γ_d is the dilaton coupling constant.

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